Unitary evolutions

(recap / high speed overview of quantum gates)

Simple example of type of quantum evolution

$$i\hbar \frac{\partial |\psi\rangle}{\partial t} = H|\psi\rangle$$ Schrödinger's Eq.

$$\Downarrow$$

$$|\psi\rangle = \underbrace{e^{-iHt}}_{U} |\psi\rangle$$

a unitary matrix $UU^{\dagger} = I$

In quantum information we often abstract away from the underlying Hamiltonian & think entirely in terms of unitary operations applied to quantum states

$\therefore$ Reversible $U^{\dagger}(U|\psi\rangle) = |\psi\rangle$

& length preserving $|\phi\rangle = U|\psi\rangle$

$\langle\phi|\phi\rangle = \langle\psi|U^{\dagger}U|\psi\rangle = \langle\psi|\psi\rangle$

2 level system
- basic building block of quantum computation

1 Single qubit gates

Basic gates: i.e. basic set of unitary operations to apply to a single qubit

classical

- Identity $|0\rangle \to |0\rangle$, $|1\rangle \to |1\rangle$ } trivial $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

- NOT $|0\rangle \to |1\rangle$, $|1\rangle \to |0\rangle$ $NOT = X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

(check: $X|0\rangle = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = |1\rangle$

$X|1\rangle = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = |0\rangle$)

Not to be confused with the Hamiltonian

- Hadamard $H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$

QUANTUM Generates non-classical states

$$H|0\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1&1\\1&-1\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix} = \frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right) = |+\rangle$$

$$H|1\rangle = |-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$$

- Evolution under Pauli Matrix

It's helpful to know how a unitary on a single qubit effects a state on a bloch sphere.

Let $\rho = \frac{1}{2}(I + \underline{r}\cdot\underline{\sigma})$

$$\rho_{out} = U\rho U^{\dagger} = \frac{1}{2}\Big(I + \underbrace{U(r\cdot\sigma)U^{\dagger}}_{\equiv s\cdot\sigma}\Big)$$

Claim: $\underline{s} = O\,\underline{r}$ where O is an orthogonal matrix

Orthogonal matrices induce (length preserving) rotations on real vectors. $\Big\{$ ie. $OO^T = O^TO = I$, $\mathrm{Real}(O) = O$

(ie. they are the real analogue of unitaries)

Proof :

To show that O is orthogonal we just need to show that $|\underline{r}| = |\underline{s}|$

To do so, we note that $\mathrm{Tr}\left((\underline{s}\cdot\underline{\sigma})^2\right) = \mathrm{Tr}\left(\left(\sum_i s_i\sigma_i\right)^2\right)$

$$= \mathrm{Tr}\left(\sum_{ij} s_i s_j \sigma_i \sigma_j \right)$$
$$= \sum_{ij} s_i s_j \delta_{ij}$$
$$= \sum_i s_i^2 = |s|^2$$

Now we note that

$$|s|^2 = \mathrm{Tr}\left((\underline{s}\cdot\underline{\sigma})^2\right) = \mathrm{Tr}\left(\left(U\,\underline{r}\cdot\underline{\sigma}\,U^\dagger\right)^2\right) = \mathrm{Tr}\left(U\,(\underline{r}\cdot\underline{\sigma})^2\,U^\dagger\right) = \sum_{ij} \mathrm{Tr}\left(r_i r_j\ \sigma_i \sigma_j\right)$$
$$= \sum r_i^2 = |r|^2$$

$$\therefore\ |\underline{s}| = |\underline{r}| \implies O \text{ is orthogonal}$$

To get a better handle on the nature of this rotation, note that:

Any unitary on a qubit can be written as

$$U = e^{i\theta\,\underline{n}\cdot\underline{\sigma}}$$

(if this isn't immediately obvious note that any unitary can be written as $U = e^{-iHt}$ and H can always be expanded in the Pauli basis)

H is now a Hamiltonian (sorry!)

which is equivalent to

$$U = \boxed{e^{-i\theta\,\underline{n}\cdot\underline{\sigma}} = \cos(\theta)\,I - i\sin(\theta)\,\underline{n}\cdot\underline{\sigma}}$$

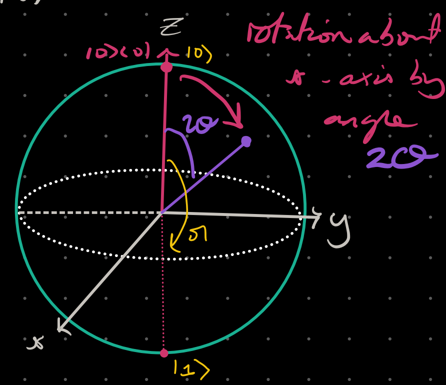
(if you've not shown this before it's worth proving it to yourself)

What is the effect of evolving $|0\rangle\langle 0|$ under $e^{-i\theta\sigma_x}$?

$$e^{-i\theta\sigma_x}\,|0\rangle = \cos\theta\,|0\rangle - i\sin(\theta)\,\sigma_x\,|0\rangle$$
$$= \cos\theta\,|0\rangle - i\sin\theta\,|1\rangle$$

eg. $$e^{-\frac{\pi}{2}\sigma_x}\,|0\rangle = \cos(\pi/2)\,|0\rangle - i\sin(\pi/2)\,|1\rangle$$
$$= -i\,|1\rangle$$

unphysical global phase



> **Exercise:** Show that $e^{-i\theta\,\underline{n}\cdot\underline{\sigma}}$ corresponds to a rotation about $\underline{n}$ vector by an angle 2θ

② Two-Qubit Gates

Examples

- Can just apply multiple copies of a gate onto systems in parallel.

  eg. $$\underbrace{X \otimes X}_{X^{\otimes 2}}\ \underbrace{|0\rangle \otimes |0\rangle}_{|0\rangle^{\otimes 2}} = X|0\rangle \otimes X|0\rangle = |1\rangle \otimes |1\rangle$$

  Alternative common notation

- More interesting: CNOT : "Apply NOT to 2nd Qubit iff 1st qubit is in state $|1\rangle$"

ie.
$$\mathrm{CNOT}\,|00\rangle = |00\rangle$$
$$\mathrm{CNOT}\,|01\rangle = |01\rangle$$
$$\mathrm{CNOT}\,|10\rangle = |11\rangle$$
$$\mathrm{CNOT}\,|11\rangle = |10\rangle$$

$\equiv$

Truth table

Before	After
00	00
01	01
10	11

11 | 10

Explicit Matrix form: $\text{CNOT} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$

(Check this really does do)

CNOT can generate entanglement.

eg. consider the sequence of gates

$$(\text{CNOT})(H \otimes I)|00\rangle = \text{CNOT}\,\underbrace{|+0\rangle}$$

non classical but unentangled state

$$= \text{CNOT}\,\tfrac{1}{\sqrt{2}}\left(|00\rangle + |10\rangle\right)$$

$$= \underbrace{\tfrac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right)}_{\text{Bell state}}$$

maximally entangled state

Bell states

$$|\Phi^\pm\rangle = \tfrac{1}{\sqrt{2}}\left(|00\rangle \pm |11\rangle\right)$$

Qubits perfectly correlated with prob ½ both 00, ½ 11

$$|\Psi^\pm\rangle = \tfrac{1}{\sqrt{2}}\left(|01\rangle \pm |10\rangle\right)$$

Qubits perfectly anti-correlated with prob ½ first qubit 0 & second 1, prob ½ first qubit 1 & second qubit 0

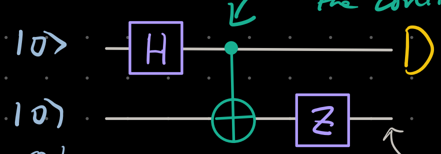
How would you prepare the other Bell states using quantum gates?

③ Drawing circuits

A tidy way of visualising a series of gates

Best understood by example

CNOT – the dot represents the control system & the ⊕ the NOT.



denotes a measurement (in computational basis unless stated otherwise)

$\equiv \quad (I \otimes Z)\ \mathrm{CNOT}\ (H \otimes I)|0\rangle$

initial state shown on the left

single qubit gates denoted as boxes on wires

each wire denotes a qubit

Be careful order of gates is inverted in circuit